Advanced Studies in Contemporary Mathematics
27 (2017), No. 2, pp. 155 - 158

www.jangjeon.or.kr
http://dx.doi.org/10.23001/ascm2017.27.2.155

New Huygens and related trigonometric and hyperbolic inequalities

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ABSTRACT: We offer new Huygens, Wilker, Cusa-Huygens, Wu-Srivastava type inequalities, which improve the existing results in the literature.

Keywords and phrases: inequalities, trigonometric functions, hyperbolic functions, Huygens inequality, Wilker inequalitiy, Cusa-Huygens inequality, Wu-Srivastava inequality
AMS Subject Classification: 26D05, 26D07, 26D99

1 Introduction

The famous Huygens' trigonometric inequality (see e.g. [1, 2], [5, 6, 7]) states that for all $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$2\sin x + \tan x > 3x \tag{1.1}$$

The hyperbolic version of inequality (1.1) has been established by E. Neuman and J. Sándor [5]:

$$2\sinh x + \tanh x > 3x \text{ for } x > 0 \tag{1.2}$$

In 1989 J. Wilker discovered another important inequality:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2, \tag{1.3}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. The hyperbolic version of this inequality has been established by L. Zhu [10].

An inequality of N. Cusa and C. Huygens (see [6] for more details regarding this result) states that

$$\frac{\sin x}{x} < \frac{\cos x + 2}{3} \tag{1.4}$$

for $x \in \left(0, \frac{\pi}{2}\right)$ (but, it can be shown that in fact, it holds for all $x > 0$.) The hyperbolic version of this inequality is due to E. Neuman and J. Sándor [5].

Finally, we mention the inequality by S. Wu and H. Srivastava [9], discovered in 2007:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \tag{1.5}$$

For a new proof of (1.5), see [1]. We note that a hyperbolic version of (1.5) appears again in the paper by Neuman–Sándor [5].

In 2012 J. Sándor [7] has discovered the following analogue of the Huygens inequality (1.1) (along with it's hyperbolic version):

$$\sin x + 4\tan\frac{x}{2} > 3x \tag{1.6}$$

It is shown also in [7] that (1.6) is a refinement of Huygens inequality (1.1) (called by the author as the second Huygens inequality).

Remark that (1.6) may be written as

$$\frac{\sin x}{x} + \frac{2\tan\frac{x}{2}}{\frac{x}{2}} > 3 \tag{1.6'}$$

We will offer a new proof to (1.6'), as well as to the following analogue of the Wilker inequality (1.3):

$$\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2 + \frac{\sin x}{x} > 2 \tag{1.7}$$

An analogue of the Wu–Srivastava inequality is:

$$\left(\frac{\frac{x}{2}}{\tan\frac{x}{2}}\right)^2 + \frac{x}{\sin x} > 2 \tag{1.8}$$

We will prove also (1.8), along with the similar hyperbolic version of this inequality.

2 Main results

For comparison of results, the following lemma will be used:

Lemma 2.1. *Let the real numbers a, b satisfy the inequality $a + 2b > 3$. Then one has also $a + b^2 > 2$.*

Proof. As $a+2b-1 > 2$, it will be sufficient to prove that $a+b^2 \geq a+2b-1$, or equivalently $b^2-2b+1 \geq 0$, i.e. $(b-1)^2 \geq 0$, which holds true. □

Lemma 2.2. *Let $x \in \left(0, \frac{\pi}{2}\right)$. Then*

$$\frac{\sin x}{x} \cdot \left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2 > 1 \tag{2.1}$$

Proof. By using $\sin x = 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$ and $\tan\frac{x}{2} = \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$, after elementary transformations, relation (2.1) becomes

$$\frac{8\sin^3\left(\frac{x}{2}\right)}{x^3\cdot\cos\left(\frac{x}{2}\right)} > 1,$$

or written equivalently:

$$\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^3 > \cos\frac{x}{2} \tag{2.2}$$

By letting $\frac{x}{2} = \theta$, (2.2) may be rewritten as

$$\frac{\sin\theta}{\theta} > \sqrt[3]{\cos\theta}, \tag{2.2'}$$

which is a famous inequality of Mitrinović–Adamović (see [3]). This finishes the proof of (2.1) □

Lemma 2.3. *Let $x > 0$. Then*

$$\frac{\sinh x}{x} \cdot \left(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\right)^2 > 1 \tag{2.3}$$

Proof. This is similar to the proof of Lemma 2.2, by remarking that the analogue of (2.2') becomes

$$\frac{\sinh\theta}{\theta} > \sqrt[3]{\cosh\theta} \tag{2.4}$$

which is another famous result by Lazarević ([3]) □

Theorem 2.1. *The Huygens type inequality (1.6'), along with its hyperbolic analogue are true.*

Proof. Put $a = \dfrac{\sin x}{x}, b = \dfrac{\tan\frac{x}{2}}{\frac{x}{2}}$. Then $a, b > 0$, and by the arithmetic mean – geometric mean inequality (for 3 numbers) we can write $a + 2b = a + b + b \geq 3\sqrt[3]{a\cdot b^2} > 3$ by the inequality $a \cdot b^2 > 1$ of Lemma 2.2.

The hyperbolic version follows on the same lines, by applying Lemma 2.3. □

Theorem 2.2. *The Wilker type inequality* (1.7)*, along with its hyperbolic analogue are true.*

Proof. Put $a = \dfrac{\sin x}{x}, b = \dfrac{\tan\frac{x}{2}}{\frac{x}{2}}$. By Theorem 2.1 one has $a + 2b > 3$. Now, by Lemma 2.1 one has also $a + b^2 > 2$, and this gives the proof of (1.7). The hyperbolic analogue of (1.7) follows by the same method, based on the hyperbolic version of (1.6') from Theorem 2.1. □

Theorem 2.3. *The Wu–Srivastava type inequality* (1.8)*, along with its hyperbolic analogue are true.*

Proof. First remark that the Cusa–Huygens inequality (1.4) may be rewritten as

$$\frac{2x}{\sin x} + \frac{x}{\tan x} > 3 \tag{2.5}$$

Now, we shall prove the following analogue of this inequality:

$$\frac{x}{\sin x} + 2 \cdot \frac{\frac{x}{2}}{\tan\frac{x}{2}} > 3 \tag{2.6}$$

In fact, this follows surprisingly from the following identity:

$$\frac{1}{\sin x} + \frac{1}{\tan x} = \frac{1}{\tan\frac{x}{2}} \tag{2.7}$$

Indeed, by putting $t = \tan\frac{x}{2}$, and using the known trigonometric relations $\sin x = \dfrac{2t}{1+t^2}$, $\tan t = \dfrac{2t}{1-t^2}$, (2.7) immediately follows by (2.7) one has

$$\frac{x}{\sin x} + 2 \cdot \left(\frac{\frac{x}{2}}{\tan\frac{x}{2}}\right) = \frac{2x}{\sin x} + \frac{x}{\tan x},$$

so (2.6) is in fact equivalent with (2.5).

Now, letting $a = \dfrac{x}{\sin x}$, $b = \dfrac{\frac{x}{2}}{\tan\frac{x}{2}}$, by Lemma 2.1, inequality (1.8) will be a consequence of (2.6).

Now, the hyperbolic analogue of (2.6) will be

$$\frac{x}{\sinh x} + 2\frac{\frac{x}{2}}{\tanh\frac{x}{2}} > 3 \tag{2.8}$$

As the hyperbolic analogue of (1.4) may be rewritten as

$$\frac{2x}{\sinh x} + \frac{x}{\tanh x} > 3, \tag{2.9}$$

the above method for the trigonometric case may be repeated, by remarking that the following analogue of (2.7) holds true:

$$\frac{1}{\sinh x} + \frac{1}{\tanh x} = \frac{1}{\tanh\frac{x}{2}} \tag{2.10}$$

If $t = \tanh\frac{x}{2}$, this follows by the known formulas,

$$\sinh x = \frac{2t}{1-t^2}, \tanh x = \frac{2t}{1+t^2}.$$

Now, using (2.9), and Lemma 2.1, the hyperbolic–version of the analogue of Wu–Srivastava inequality follows as well. □

Remark 1. *Inequality* (1.7) *has been discovered in another form by E. Neuman [4]:*

$$\frac{2x}{\sin x} < 1 + \frac{\sin x}{x} \cdot \left(\frac{2}{1+\cos x}\right)^2$$

Indeed, as $\dfrac{\cos x + 1}{2} = \cos^2\dfrac{x}{2}$*, and* $\sin x = 2\sin\dfrac{x}{2} \cdot \cos\dfrac{x}{2}$*, after some elementary transformations, this inequality becomes in fact* (1.8).

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